Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021-8928(00)00132-5

J. Appl. Maths Mechs, Vol. 64, No. 6, pp. 1005–1014, 2000

0021-8928/00/\$—see front matter

PROBLEMS OF MAXIMIZING THE COMPLIANCE OF CURVILINEAR RODS†

I. N. DASHEVSKII and V. E. SHKLOVER

Moscow

email: dash@ipmnet.ru

(*Received* 6 *December* 1999)

A number of problems of finding the shape of a thin curvilinear rod (the support element of an artificial lens) of constant cross-section and specified length with its ends at specified points and under specified loading conditions with maximum compliance for characteristic types of end restraint and loading are considered. It is shown that the boundary-value problem arising for the non-linear Euler equation may have a set (possibly denumerable) of solutions, one of which gives the absolute maximum compliance, and the others the local maxima. The problem is analysed in detail, analytical solutions are obtained and the corresponding shapes are constructed in a number of important cases.

In a number of problems related to structures containing thin curvilinear rods as, support elements (SEs) the need arises to find the shape of the SE that has maximum compliance. This is required, in particular, in problems of optimizing the structures for an intracapsular artificial lens [1]. Below, the problem of maximizing the compliance is examined as a variational problem on the bending of an SE as a curvilinear rod with constant diameter and cross-section moment of inertia. In problems examined earlier [2], concerning rods of minimum compliance, the required curve is uniquely defined. A detailed proof of the adopted formulation and a discussion of other approaches to describe the mechanics of the "capsule-artificial lens" system were given in earlier publications [1, 3–5].‡

1. FORMULATIONS OF THE PROBLEM. FORMULAE FOR THE COMPLIANCE

For linear systems, the relation between generalized forces and the corresponding generalized displacements is linear: $u_1 = \Pi_{ij} Q_j$, where $\| \Pi_{ij} \|$ is the matrix of the compliances [6, p. 457], or the "matrix of the external compliance of the system" [7, p. 58], or the "matrix of the coefficients of influence" [8, p. 159]. Below, in all cases we will examine only one generalized displacement as a result of the action of one generalized force, and therefore we will use the above relation in the form $u = \Pi Q$ or $\Pi = u/Q$, specifying all the more precisely what a compliance Π, a generalized force Q and a generalized displacement u will be involved in each specific case. Then, the requirement of maximum compliance can be written in general form as

$$\Pi = u/Q \to \max \tag{1.1}$$

where u is the characteristic displacements of points of the SE, and Q represents the loads applied to the SE or the loads that arise reactively from the external structure.

From the viewpoint of a qualitative analysis and of obtaining effective solutions, it is more convenient to deal with point loads (see Fig. 1; for an open SE its lower part is shown by the continuous curve, and for a closed SE it is shown by the dashed curve; ES stands for external structure. However, physically, the contact area of the SE with the external structure has finite dimensions (Fig. 2), and Q in (1.1) is the pressure q in the contact area (it is convenient to assume this to be constant). With a non-zero contact area, its shape and dimensions in the optimization formulation, up to Section 6, will be assumed to be specified and (in accordance with the ophthalmological subtext: an SE resting on the circular capsule of an artificial lens) to have the form of the arc of a circle.

†*Prikl. Mat. Mekh.* Vol. 64, No. 6, pp. 1046–1056, 2000.

‡See also also DASHEVSKII, I. N. and SHKLOVER, V. E., Some problems related to the mechanics of an artificial lens. Preprint No. 355, Inst Problem Mekh. Akad. Nauk SSSR, Moscow, 1988, DASHEVSKII, I. N. and SHKLOVER, V. E., *Intracapsular artificial lens and curvilinear rods of maximum compliance.* Preprint No. 398, Inst. Problem Mekh. Akad. Nauk SSSR Moscow, 1989.

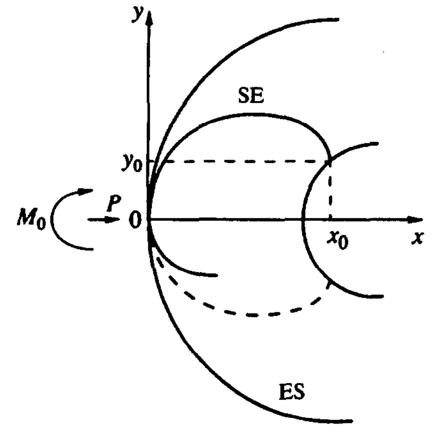

Fig. 1

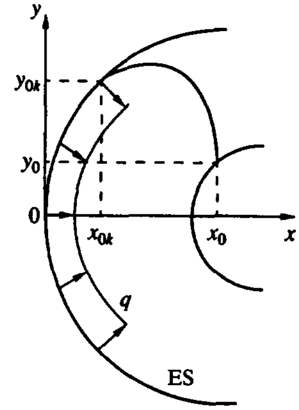

Fig. 2

If u is the generalized displacement at a point a in any direction under the action of the generalized force Q applied at a point b (generally speaking, $b \neq a$) or distributed over a finite region, then, for a plane rod system under bending, which, in the general case, in statically indeterminate, the formula for the compliance using Mohr integrals can be written in rectangular Cartesian coordinates x, y in the following form ([18, p. 161; 9, p. 199], etc.)

$$\Pi = \frac{u}{Q} = \frac{1}{EJ}\int_0^{x_0} M_{1a} M_* \sqrt{1+y'^2}\,dx, \quad M_* = M_{1b} + \lambda_1 M_{1c_1} + \ldots + \lambda_n M_{1c_n} \tag{1.2}$$

where E is Young's modulus, J is the moment of inertia, x_0 is the abscissa of the end of the SE, $y(x)$ is the shape of the SE, M_{1a} is the distribution of the moment due to a unit force factor acting at the point a in the direction of displacement, M_{1b} is the same, but at the point b in the direction of application of the force, $\lambda_1, \ldots, \lambda_n$ are the constraint reactions at points $c_1, c_2, \ldots, c_n$, generated by a unit force factor Q, M_{1c} represents the distributions of the moments generated by unit constraint reactions at the point sc (for a statically determinate system $\lambda_i = 0$ and $M_* = M_{1b}$). The constraint equations expressing the absence of displacements at the points $c_1, c_2, \ldots c_n$ have the form

$$\int_0^{x_0} M_{1c_i} M_* \sqrt{1+y'^2}\,dx = 0, \quad i = 1, \ldots, n \tag{1.3}$$

If the directions of the force and displacement and also the points a and b coincide, then, by replacing a by b in Eq. (1.2) and multiplying each of equalities (1.3) by λ_i and adding them all up, instead of (1.2) we obtain

$$\Pi = \frac{1}{EJ}\int_0^{x_0} M_*^2 \sqrt{1+y'^2}\,dx \tag{1.4}$$

Closure of the formulation of the optimization problem necessitates specifying the boundary conditions (the coordinates of the rod ends) and the condition of a fixed length of the SE

$$y(0) = 0, \quad y(x_0) = y_0, \quad \int_0^{x_0} \sqrt{1+y'^2}\,dx = l \tag{1.5}$$

2. MAXIMIZING THE COMPLIANCE FOR ANGULAR DISPLACEMENT FOR AN OPEN SE

We will first consider the simplest model case. We will introduce a system of coordinates and notation as indicated in Fig. 1; here, u will denote the angular displacement of the open SE (the angle through which the tangent to the SE turns), for example at point O with a point force P acting on the SE at this point along the x axis (an examination of the action of a uniformly distributed normal pressure q (Fig. 2) is similar to that carried out in the present section).

We will use an expression for Π in the form of (1.2). In the given case, $a = b$, $M_{1a} = 1$ and $M_* = M_{1b} = y(x)$, which gives the optimization problem in the form

$$EJ\Pi = \int_0^{x_0} y(x)\sqrt{1+y'^2}\,dx \to \max \tag{2.1}$$

under conditions (1.5).

Problem (2.1), (1.5) is an isoperimetric problem of the calculus of variations, mathematically equivalent (apart from an unimportant max $\leftrightarrow$ min replacement) to the classical Bernoulli problem on the shape of a heavy thread (chain or rope) in a gravitational field with a solution in the form of a catenary [10, p. 390].

According to the theorem of displacement reciprocity ([8, p. 159], [9, p. 214]), the solution obtained is also the solution of the problem of optimizing the compliance for a linear displacement along the Ox axis when the moment M_0 is applied to the SE at the point O.

3. MAXIMIZING THE COMPLIANCE OF AN OPEN SE FOR A LINEAR DISPLACEMENT. A POINT FORCE

The formulation of this section corresponds to Fig. 1, where u denotes the linear displacement of the SE at the point O along the direction of the force $Q = P$. Here, in (1.2) $M_* = M_{1b}$, $a = b$ and the functional being optimized acquires the form

$$EJ\Pi = \int_0^{x_0} y^2(x)\sqrt{1+y'^2}\,dx \to \max \tag{3.1}$$

under conditions (1.5).

Introducing the Lagrangian multiplier μ and investigating, for an unconditional extremum, the functional

$$\int_0^{x_0} (y^2+\mu)\sqrt{1+y'^2}\,dx \to \max \tag{3.2}$$

we obtain Euler's equation

$$\frac{y''}{1+y'^2} = \frac{2y}{y^2+\mu} \tag{3.3}$$

To clarify certain general features of the solution, initially we will consider the limit case of a shape which deviates slightly from the x axis, formally: $y^2(x) \ll |\mu|$ and $y'^2 \ll 1$. Here Eq. (3.3) and the last condition of (1.5) degenerate to

$$y'' = \frac{2y}{\mu}, \quad \int_0^{x_0} y'^2 dx = 2(l-x_0) \tag{3.4}$$

Solving Eq. (3.4), taking account of the first two conditions of (1.5), and then using relation (3.1), we obtain for the shape and the compliance

$$y(x) = A\sin\left(\omega\frac{x}{x_0}\right), \quad \Pi = \frac{A^2x_0}{2EJ}\left(1-\frac{\sin 2\omega}{2\omega}\right) \tag{3.5}$$

The integration constants (the frequency ω and the amplitude A) are defined by the formulae

$$\frac{\sin^2\omega}{\omega^2} = \frac{y_0^2}{4x_0(l-x_0)}\left[1+\frac{\sin 2\omega}{2\omega}\right], \quad A = \pm\frac{2}{\omega}[x_0(l-x_0)]^{1/2}\left(1+\frac{\sin 2\omega}{2\omega}\right)^{-1/2} \tag{3.6}$$

If the dimensionless parameter $\varepsilon = y_0(2\sqrt{x_0(l-x_0)}) \ll 1$ (which somewhat reinforces the initial assumptions), then, from the second relations of (3.6) we obtain that, if the roots ω_m are not too large, such that $\omega_m\varepsilon \ll 1$, then $|\sin\omega_m| \ll 1$. Hence, $\omega_m = \pi m + \omega \approx \pi m$, where $|\omega| \ll 1$, $m = 1, 2, \ldots \ll 1/(\pi\varepsilon)$, which corresponds to the solution $y_m(x)$ with approximately m half-waves (solutions with

$m = 0$ when $\varepsilon \ll 1$ are impossible, since then $|\omega_m| = |\omega| \ll 1$ and the ratio $\sin \omega_m/\omega_m$ would be of the order of unity and not of the order $\varepsilon \ll 1$; solutions with $m < 0$, however, can be rejected, since, in view of the fact that $\omega_m \approx \pi m$, they only change the sign of $y_m(x)$ in (3.5), which is already taken into account by the $\pm$ signs in the second relation of (3.6) for A). For the amplitude, ignoring the term $\sin 2\omega/(2\omega)$ in the second relation of (3.6), we obtain $A_m = \pm 2[x_0(l - x_0)]^{1/2}(\pi m)$. The expressions for the optimum shape and compliance in this case can be written explicitly.

$$y_m(x) = \pm\frac{\sqrt{2x_0(l-x_0)}}{\pi m}\sin \pi m\frac{x}{x_0}, \quad \Pi_m = \frac{2x_0^2(l-x_0)}{\pi^2 m^2 EJ}$$

In view of the presence of the integer-valued parameter m, the solution of the problem, generally speaking, is multivalued, and here to each permissible m there corresponds a solution with m half-waves.

The equality $y_0 - 0$ implies that $\varepsilon = 0$, and the explicit formulae obtained for $\omega_m, A_m, y_m(x)$ and Π_m become accurate according to (3.6) and (3.5) within the framework of the initial assumptions, while the number of solutions becomes denumerable ($m = 1, 2, \ldots$).

Returning to Euler's equation (3.3), in the general case of large $y(x)$, we obtain its first integral in the form

$$p = \frac{dy}{dx} = \pm\sqrt{\left(\frac{y^2+\mu}{c}\right)^2 - 1} \tag{3.7}$$

where, to be specific, it is assumed that the integration constant $c > 0$.

The application of the Legendre condition (checking the sign of the second derivative with respect to y') to (3.2) indicates that the type of the extremum is determined by the sign of $y^2 + \mu$. From (3.7) it follows that $y^2 + \mu$ does not vanish in the segment $x \in [0, x_0]$, i.e. it retains a constant sign. If $y^2 + \mu > 0$, the solution gives a minimum of functional (3.2) provided that $y(0) = 0$; in this case it is monotonic and is uniquely defined [2]. The expression $y^2 + \mu < 0$, which changes the situation fundamentally, corresponds to the case of the maximum.

Separating the variables x and y, we rewrite relation (3.7) in the form

$$\frac{dy}{\sqrt{(-y^2-\mu-c)(-y^2-\mu+c)}} = \pm\frac{dx}{c} \tag{3.8}$$

Since $y^2 + \mu < 0$, we have $\mu < 0$. From (3.8) it follows that $-\mu - c > 0$ and $|y| \leqslant \sqrt{-\mu - c}$. Therefore, from y, μ and c it is possible to change to Θ, k and ω

$$y = \sqrt{-\mu-c}\sin\Theta, \quad \Theta(0) = 0, \quad k = \sqrt{\frac{-\mu-c}{-\mu+c}}, \quad 0 < k < 1, \quad \omega = \frac{x}{c}\sqrt{-\mu+c} \tag{3.9}$$

We adopt the condition $\Theta(0) = 0$ taking into account the first condition of (1.5) and assume that $x_0 > 0$ (the case where $x_0 = 0$ is degenerate and, after investigating the general behaviour pattern of the solutions, is naturally obtained as the limit as $x_0 \to 0$).

Changing in Eq. (3.8) to the variables Θ, k and ω and integrating it under the condition $\Theta(0) = 0$, we obtain $\Theta = \pm \operatorname{am}(\omega x/x_0, k)$, where $\operatorname{am}(u, k)$ is the Jacobi elliptic function, and the "plus" and "minus" signs correspond to two different solutions.

Returning, using (3.9), from Θ to y, we have

$$y(x) = \pm\frac{2kx_0}{\omega(1-k^2)}\operatorname{sn}\left(\omega\frac{x}{x_0}, k\right) = \pm\sqrt{-\mu-c}\,\operatorname{sn}\left(\frac{\sqrt{-\mu-c}}{c}x, \sqrt{\frac{-\mu-c}{-\mu+c}}\right) \tag{3.10}$$

$$\operatorname{sn}(u,k) \equiv \sin(\operatorname{am}(u,k)), \quad u = \int_0^{\operatorname{am} u}\frac{dt}{\sqrt{1-k^2\sin^2 t}}.$$

Thus, the solution is a function resembling a sinusoid with a frequency-dependent amplitude.

Substituting expression (3.10) into (3.1) and using certain properties of the Jacobi elliptic functions [11, pp. 791, 792 and 47–51], after some reduction we obtain an expression for the compliance in the form

$$\Pi = \frac{8x_0^3}{3EJ}\frac{1}{\omega^3}\frac{k^2}{(1-k^2)^3}\left\{\frac{1-k^2}{2k^2}\left[\frac{(1+k^2)(1+\bar{l})}{2}-1\right]\omega - \operatorname{sn}\omega\operatorname{cn}\omega\operatorname{dn}\omega\right\},\quad \bar{l}=\frac{l}{x_0} \tag{3.11}$$

To determine the constants k and ω, from the second condition of (1.5) we obtain

$$\operatorname{am}(\omega, k) = (-1)^m \arcsin\frac{\pm y_0\omega(1-k^2)}{2kx_0} + \pi m \tag{3.12}$$

where m is an integer equal to the rounded-off number of half-waves which fit in the segment $[0, x_0]$. From the relations $c > 0$, $-\mu > 0$ and $x_0 > 0$, it follows from (3.9) that $\omega > 0$ and $\operatorname{am}(\omega, k) > 0$, and from (3.12) we establish that $m \geqslant 0$, and here if $\pm y_0 \leqslant 0$, then $m \geqslant 1$. Changing from (3.12) to the relation $F(f_1, k) = F(f_2, k)$, where f_1 and f_2 are respectively the right- and left-hand sides of (3.12), while $F(\omega, k)$ is the normal Legendre elliptic integral of the first kind [12, p. 753], and taking account of the fact that $F[\operatorname{am}(\omega, k), k] = \omega$. we obtain

$$\pm(-1)^m F\left(\arcsin\frac{y_0\omega(1-k^2)}{2kx_0}, k\right) + 2m\mathbf{K}(k) = \omega \tag{3.13}$$

where $\mathbf{K}(k)$ is the complete normal Legendre elliptic integral of the first kind [12, p. 757 and thereafter].

Then, from the final condition of (1.5), taking into account relations (3.7) and (3.10) and the properties of the Jacobi elliptic functions, after some reduction we obtain

$$E(\operatorname{am}(\omega, k), k) = \frac{(l+x_0)\omega(1-k^2)}{2x_0} = \omega\frac{1-k^2}{2}(1+\bar{l})$$

where $E(\varphi, k)$ is the normal Legendre elliptic integral of the second kind [12, p. 753].

Substituting (3.12) into the last relation, we arrive at

$$\pm(-1)^m E\left(\arcsin\frac{y_0\omega(1-k^2)}{2kx_0}, k\right) + 2m\mathbf{E}(k) = \frac{(l+x_0)\omega(1-k^2)}{2x_0} \tag{3.14}$$

where $\mathbf{E}(k) = E(\pi/2, k)$ is the complete normal Legendre elliptic integral of the second kind [12, p. 757 and thereafter].

Relations (3.13) and (3.14) give two equations for determining the two unknown integration constants k and ω. In view of the presence in (3.13) and (3.14) of an undetermined integral-valued parameter m, the solution of maximization problem (3.2) under conditions (1.5), generally speaking, is multivalued (several extremals and the corresponding local maxima). To find the global maximum, it is necessary to find the compliance for each of the local maxima (i. e. for each permissible m) and select the greatest of them.

When $y_0 = 0$, relations (3.13) and (3.14) become

$$\omega = 2m\mathbf{K}(k) \tag{3.15}$$

$$\mathbf{E}(k) / \mathbf{K}(k) = (1 - k^2)(1 + l/x_0)/2 \tag{3.16}$$

Equation (3.16) has a unique solution $k(l/x_0)$ [12, p. 757] and thereafter], according to which relation (3.15) determines a denumerable set of frequencies ω_m for all $m = 1, 2, \ldots$

Substituting expression (3.15) into (3.10) and (3.11), and taking account of the fact that in this case $sn\omega = sn(2m\mathbf{K}(k)) = 0$, we obtain for the solution and the compliance, respectively

$$y_m(x) = \pm\frac{kx_0}{m(1-k^2)\mathbf{K}(k)}\operatorname{sn}\left[2m\mathbf{K}(k)\frac{x}{x_0}, k\right],\quad x\in[0, x_0] \tag{3.17}$$

$$\Pi_m = \frac{x_0^3}{3EJm^2}\frac{1}{(1-k^2)^2\mathbf{K}(k)^2}\left[\frac{(1+k^2)(1+\bar{l})}{2}-1\right] = \frac{\Pi_1}{m^2}$$

The global maximum of the compliance corresponds to $m = 1$.

The solutions obtained when $y_0 = 0$ (for the plus sign in (3.17)) are shown in Fig. 3 by the solid lines. When $m = 1$, the solution consists of a single half-wave, and in the entire segment $x \in [0, x_0]$ it is convex (for the plus sign) and within it does not vanish. When $y_0 \neq 0$ and $m \gg 1$, (3.17) is an approximate solution if a certain neighbourhood of the end (x_0, y_0) is ignored.

If $y_0 = 0$, then m is exactly equal to the number of half-waves, but if $y_0 \neq 0$, then m is the result of rounding-off this number: slightly less than, or more than, m half-waves fit in the segment, and the difference does not exceed a quarter-wave (Fig. 3, the dashed lines).

Investigation of relations (3.10), (3.13) and (3.14), taking into account relations (3.9) and the signs of the parameters, leads to the general behaviour pattern of the solutions which can be presented graphically if x_0, $l > x_0$ is given, if y_0 is "freed" and if the evolution of the solution as a function of ω is followed. The value $\omega = 0$ corresponds to a solution which is a segment joining the origin of coordinates to the point $(x_0, \pm(l^2 - x_0^2)^{1/2})$. As ω increases, this segment begins to bend, retaining its length l and sliding, with its right-hand end, along the vertical $x = x_0$. When ω reaches a value of $2\mathrm{K}(k)$, where k is the root of Eq. (3.16), Eq. (3.15) is satisfied for the first time, which corresponds to the case where, in the segment $[0, x_0]$, just one half-wave of the solution is located. Further increase in ω for a fixed length l leads to the formation of an increasing number of folds/half-waves and is accompanied by a reduction in the amplitude of the solution and attenuating oscillations of its right-hand end about the point $(x_0 y_0 = 0)$ according to the law

$$y_0 = \pm 2kx_0 \operatorname{sn}(\omega, k)/[\omega(1 - k^2)]$$

Here, every time ω passes through $2m\mathrm{K}(k)$, the quantity y_0 passes through zero (determined by the zeros of the function $sn(\omega, k)$) and the SE shape passes a configuration with an integer number m of half-waves.

In the general case, for given $y_0 \neq 0$, we can obtain a bilateral estimate for m (see the references)

$$[1 - \operatorname{sign}(\pm y_0)]/2 \leqslant m \leqslant [(1 + x_0)/|y_0| + 1]/2$$

and a more accurate upper estimate

$$m \leqslant (l^2 - x_0^2)^{1/2}(2 \mid y_0 \mid) + 1$$

However, the greater the value of m, the lower the required compliance. From this it follows that the global maximum, i.e. the solution of problem (1.1), is obtained when $m = 0$ or $m = 1$.

It is interesting to compare the compliance $\Pi_1(l)$ of the optimum shape (for simplicity, with $y_0 = 0$) with the compliance of model shapes of the form Λ and Π with the same l and x_0. The expressions for Π_Λ and Π_Π and asymptotically for Π_1 as $\bar{l} \to 1$ and $\bar{l} \to \infty$ have the following respective forms

$$\Pi_\Lambda = \frac{x_0^3 \bar{l}(\bar{l}^2 - 1)}{12EJ}, \quad \Pi_\Pi = \frac{x_0^3(\bar{l} - 1)^2(\bar{l} + 2)}{12EJ}, \quad \Pi_1 \underset{\bar{l} \to 1}{\approx} \frac{2x_0^3(\bar{l} - 1)}{\pi^2 EJ}, \quad \Pi_1 \underset{\bar{l} \to \infty}{\approx} \frac{x_0^3 \bar{l}^3}{12EJ}$$

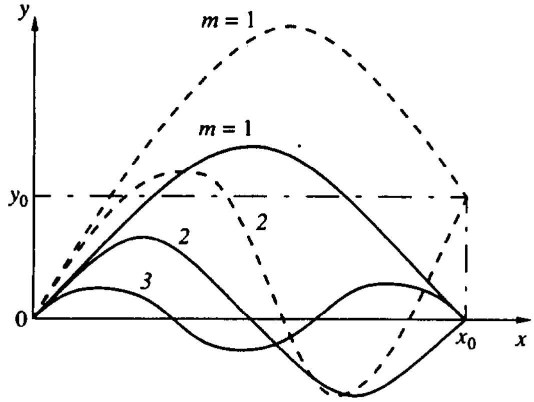

Fig. 3

$$\lim_{\bar{l}\to 1}\frac{\Pi_1}{\Pi_\Lambda}=\frac{12}{\pi^2},\quad \lim_{\bar{l}\to 1}\frac{\Pi_1}{\Pi_\Pi}=\infty,\quad \lim_{\bar{l}\to\infty}\frac{\Pi_1}{\Pi_\Lambda}=\lim_{\bar{l}\to\infty}\frac{\Pi_1}{\Pi_\Pi}=1$$

Hence, unlike the case when $\bar{l}\to 1$ ($l\to x$), when $\bar{l}\to\infty$ the compliances Π_1, Π_Λ and Π_Π are equivalent and proportional to $\bar{l}^3$ (see Fig. 4, where the lines 1, 2 and 3 correspond to $\ln(EJ\Pi_1)$, $\ln(EJ\Pi_\Lambda)$ and $\ln(EJ\Pi_\Pi)$).

4. A DISTRIBUTED LOAD

We will consider the same problem for the case of a uniform load distribution (Fig. 2) so that $Q = q$ in (1.1). As in Section 1, we will assume the contact part of the SE to be the arc of a circle of specified radius R and angular dimension φ_0, and only the contact part of the SE $y(x)$, $x\in[x_{ok}, x_0]$, where x_{ok} $R[1-\cos(\varphi_0/2)]$ is the abscissa of the boundary of the contact area, is to be determined.

By symmetry, the overall force acting on the SE from the capsule is directed along the Ox axis, and the overall moment taken about any point on this axis is equal to zero. It is therefore more convenient to optimize the compliance Π with respect to the displacement u in the middle of the contact area in the direction of the corresponding radius.

Substitution of the corresponding expressions for M_{1a} and M_{1b} into relations (1.2) and (1.5) leads to the optimization problem

$$EJ\Pi = EJ\Pi_{ok} + 2R\sin\frac{\varphi_0}{2}\int_{x_{ok}}^{x_0} y^2\sqrt{1+y'^2}\,dx \to \max$$

$$y(x_{ok}) = y_{ok} = R\sin(\varphi_0/2),\quad y(x_0)=y_0,\quad \int_{x_{ok}}^{x_0}\sqrt{1+y'^2}\,dx = l - R\varphi_0$$

where the compliance of the contact part of the SE, Π_{ok}, is a known function of r and φ_0.

Solving this problem in the same way as in Section 3, we obtain for the contact part of the SE

$$y(x) = \pm\frac{2kx_0}{\omega(1-k^2)}\operatorname{sn}\left(\omega\frac{x-x^*}{x_0},k\right),\quad x\in[x_{ok},x_0]$$

$$x^* = x_{ok} + \frac{x_0}{\omega}\left\{\mathbf{K}(k) - \left\{\pm\left[F\left(\arcsin\frac{y_{ok}\omega(1-k^2)}{2kx_0},k\right)+\mathbf{K}(k)\right]\right\}\right\}$$

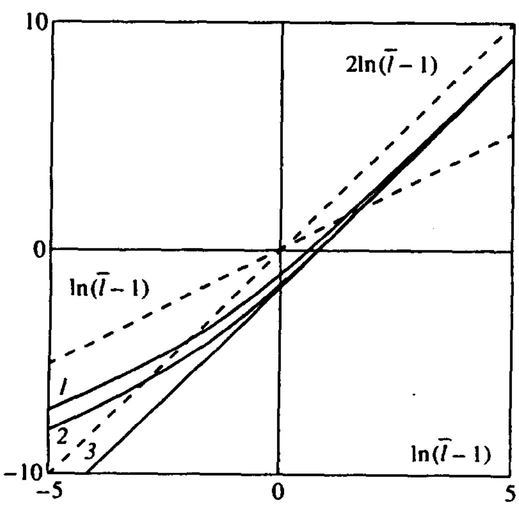

Fig. 4

We will not give the conditions for integration constants k and ω and the formula for the compliance, again containing the integer-valued parameter m, in view of their complexity (see the references). When $y_0 = y_{ok}$ (the line joining the ends of the non-contact part of the SE is parallel to the Ox axis) for even m, or when $y_0 = -y_{ok}$ for odd m, the calculations are simplified somewhat and, for constants k and ω, give the system of equations

$$\omega = 2m\frac{x_0}{x_0 - x_{ok}}\mathbf{K}(k), \quad \frac{\mathbf{E}(k)}{\mathbf{K}(k)} = \frac{1-k^2}{2}\left[1 + \frac{l - R\varphi_0}{x_0 - x_{ok}}\right]$$

In this case, in the segment $x \in [x_{ok}, x_0]$, an integer even (or odd) number of half-waves fits for the corresponding part of the solutions and, like the results in Section 3, when $y_0 = 0$, we have $\omega_m = m\omega_1$ and $k_m = k_1$. However, here the number m is limited by the inequality

$$|y_{0m}| = |y_0| \leq |y_{\max}| = 2kx_0/[m\omega_1(1-k^2)]$$

The reason is that the solution fluctuates as before about the x axis and not about the line $y_0 = y_{ok}$. Since, as m increases the amplitude of the solution decreases and the solution is increasingly closely pressed towards the x axis, starting from a certain m, it is impossible to satisfy the conditions on the edges with specified final $y_{ok} = |y_0|$. The required compliance is calculated by the formula $\Pi_m = \Pi_{ok} + \Pi_{nk}/m^2$, where Π_{nk} is the compliance of the non-contact part of the SE.

Note that the solution of the problem of maximizing the compliance can be obtained in explicit form, generally speaking, only for a displacement direction coinciding with the direction of the overall external force acting on the SE. In the opposite case, the corresponding Euler equation cannot be integrated in explicit form (this is possible only in the limit where the length of the SE is close to the length of the segment joining is ends).

5. MAXIMIZING THE COMPLIANCE OF A SYMMETRICAL CLOSED SE (FIG. 5)

For convenience, the force applied to the SE will be denoted by $2P$; then $\Pi = u/(2P)$.

Selecting the upper half of the SE as the main system (it accounts for half of the applied force, i.e., a force P), replacing the bottom half with two reactions – with a force P_{0y} and a moment M_0, which are proportional to the force P with unknown coefficients λ_1 and λ_2: $P_{0y} = \lambda_1 P$, $M_0 = \lambda_2 P$ (by symmetry, there is no corresponding component P_{0x} along the O_x axis, and using formula (1.4), we obtain the problem of optimizing the functional

$$2EJ\Pi = \int_0^{x_0} (y - \lambda_1 x - \lambda_2)^2\sqrt{1 + y'^2}\,dx \to \max \tag{5.1}$$

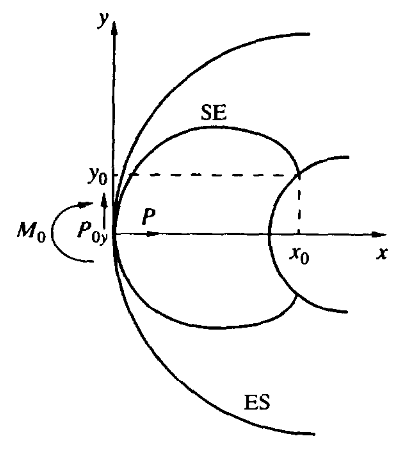

Fig. 5

under conditions (1.5), and also the conditions for there to be no displacement along the Oy axis and no rotation

$$\int_0^{x_0} x(y-\lambda_1 x-\lambda_2)\sqrt{1+y'^2}\,dx=0,\quad \int_0^{x_0}(y-\lambda_1 x-\lambda_2)\sqrt{1+y'^2}\,dx=0 \tag{5.2}$$

Introducing conditions (5.2), into the functional (5.1) using Lagrange multipliers, it can be shown that the requirement that the derivatives of the required extremal with respect to λ_1 and λ_2 should vanish means that these Lagrange multipliers are equal to zero. Therefore, by introducing the former multiplier μ, corresponding to the third condition of (1.5), we obtain the problem of searching for an unconditional extremum of the functional

$$\int_0^{x_0}\{(y-\lambda_1 x-\lambda_2)^2+\mu\}\sqrt{1+y'^2}\,dx\to\max$$

under conditions (1.5) and (5.2).

Replacement of the variables by means of an orthogonal transformation

$$x_1=\frac{x+\lambda_1 y}{\sqrt{1+\lambda_1^2}},\quad y_1=\frac{y-\lambda_1 x-\lambda_2}{\sqrt{1+\lambda_1^2}}\quad ((0,0)\to(x_{1c},y_{1c}),\ x_{1c}=0)$$

enables us to reduce the integral being maximized to a form similar to (3.2), and to obtain the solution of the corresponding optimization problem in the form (Fig. 6)

$$y_1(x_1)=\pm\frac{2k(x_{10}-x_{1c})}{\omega(1-k^2)}\operatorname{sn}\left(\omega\frac{x_1-x_1^*}{x_{10}-x_{1c}},k\right),\quad x_1\in[x_{1c},x_{10}]$$

$$x_1^*=x_{1c}+\frac{x_{10}-x_{1c}}{\omega}\left\{\mathbf{K}(k)\mp\left[F\left(\arcsin\frac{y_{1c}\omega(1-k^2)}{2k(x_{10}-x_{1c})},k\right)+\mathbf{K}(k)\right]\right\}$$

The conditions for determining the constants k, ω, λ_1 and λ_2, and also the formulae for the compliance, are omitted in view of their complexity (see the references). The lower part of the SE is symmetrical with the upper part about the Ox axis.

As above, the complete integrals of Euler's equation can be obtained for a SE with hinged ends, and also for a number of other model structures of a closed SE.

6. THE REQUIREMENTS OF SMOOTHNESS AND INSCRIBABILITY OF THE REQUIRED SHAPES IN THE SPECIFIED REGION

An attempt to use directly the optimum shapes obtained in this investigation in artificial lens structures encounters a number of problems. The first is the non-smoothness of these shapes at the point where

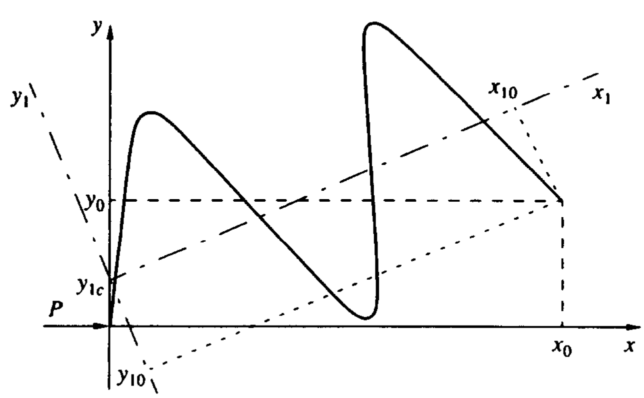

Fig. 6

they meet the capsule of the artificial lens: generally speaking, the solutions for the SE yield shapes which either enter the capsule at the point O (for an open SE, this is its free end, and for a closed SE its middle) or (Section 4) which have a break on the boundary of the contact area of the SE with the capsule.

It seems that, for small $l \sim x_0$, solutions in the class of smooth functions in the given sense do not generally exist. However, in artificial lens models that are actually used, to obtain adequate compliance, the length of the SE, l, should be (and is in fact selected to be) fairly long ($l \sim (x_0 + R)$). This makes it necessary to satisfy another natural additional condition for the SE of the artificial lens: the requirement of 'inscribability" of the required SE shape in a certain region (assuming the displacements are small, this is roughly in the annular region between the circular capsule and circular artificial lens). At the same time, the somewhat unexpected possibility is opened up of correcting this defect of non-smoothness just in the range of long lengths l, which is of greatest practical importance.

The pattern of evolution of the solution as l increases will qualitatively be as follows. For small $l \sim x_0$, the corresponding extremal fits within the specified region and the solution does not reach its external boundary, with the exception of the point O. As l increases, the amplitude of the solution increases up to the instant of contact of the extremal of the capsule circle, after which, when l increases further from the point of contact, a region of overlap is formed and continues to grow until, at a certain $l \sim (x_0 + R)$, the entire "near-capsule" part of the solution lies on the capsule. From this instant, all the breaks and "non-smoothnesses" of the solution also disappear.

Obtaining a smooth solution of this kind can be interpreted as the result of taking into account the requirement of inscribability for fairly long l, which leads to the problem of an extremum with constraints on the shapes required in the form of inequalities. As is well known [12, pp. 352 and 353], when solving such problems in the class of smooth functions, the extremum can be reached only on curves consisting of arcs of extremals (satisfying Euler's equation) and parts of the boundary of the permissible region ("the capsule plus the surface of the artificial crystalline lens"). In this case, at the points of transition of extremals on the boundary, they have a common tangent and the position of these points is determined from the smoothness conditions.

However, it is possible to regard obtaining such a solution as the result of correcting the non-smoothness defects of the initial solution, for which an additional condition needs to be introduced into the initial formulation: the requirement that the rod (SE) should be tangent to the capsule at points where they meet. However, the simple addition of such a condition will generally result in the problem not having a solution as a boundary-value problem for an ordinary second-order differential equation (Euler's equation) with three boundary conditions (the first two conditions fix the position of the ends of the rod). To obtain a correct formulation, it is necessary, at the same time as introducing the additional condition, to introduce an additional free parameter, the value of which will be determined automatically when solving the problem. For example, the length of the SE, l, can be adopted as such a parameter. By "freeing" the length l, and increasing it, we obtain precisely the same pattern of evolution of the solution as above.

REFERENCES

1. DASHEVSKII, I. N. and SHKLOVER, V. E. The intracapsular artificial lens and the inverse problem of the theory of flexible rods. *Izv. Akad, Nauk SSSR. MTT*, 1990, 5, 158–165.
2. BANICHUK, N. V., Determination of the optimum shapes of elastic curvilinear rods. *Izv. Akad. Nauk SSSR. MTT*, 1975, 6, 124–133.
3. ANAN'YEVSKII, I. M., GOL'DSHTEIN, R. V., DASHEVSKII, I. N. *et al.*, some problems of the mechanics of an intracapsular artificial lens. In *Medical Biomechanics*, Vol. 1, *Proceedings of the International Conference on "Progress of Biomechanics in Medicine"* (Riga, 1986), Latvian Ministry of Health, Riga, 1986, 28–33.
4. SAAKYAN, Yu. A., TSATURYAN, A. K., SEMENOV, P. A. *et al.*, Modelling of the biomechanical behaviour of the capsule of the lens of the eye during the implantation of intraocular lenses. *In Medical Biomechanics*, Vol. 1, *Proceedings of the International Conference on "Progress of Biomechanics in Medicine"* (Riga, 1986), Latvian Ministry of Health, Riga, 1986, 337–341.
5. ANAN'YEVSKII, I. M. and CHERNOUS'KO, F. L., Investigation of the mechanics of support elements of an artificial lens. *Izv. Akad, Nauk SSSR. MTT*, 1988, 4, 175–180.
6. TIMOSHENKO, S. P. and GERE, J. M. *Mechanics of Materials*. Van Nostrand Reinhold Company, New York, 1972.
7. RZHANITSYN, A. R., *Structural Mechanics*. Vysshaya Shkola, Moscow, 1991.
8. RABOTNOV, Yu. N., *Solid Mechanics*. Nauka, Moscow, 1979.
9. FEODOS'YEV, V. I., *Strength of Materials*. Nauka, Moscow, 1986.
10. EL'SGOL'TS, L. E., *Differential Equations and Calculus of Variations*, Nauka, Moscow, 1965.
11. PRUDNIKOV, A. P., BRYCHKOV, Yu. A. and MARICHEV, O. I., *Integrals and Series. Additional Chapters*, Nauka, Moscow, 1986.
12. KORN, G. A. and KORN, T. M., *Mathematical Handbook for Scientists and Engineers*, McGraw-Hill, New York, 1968.

Translated by P.S.C.